

# LATTICE BOSONIZATION

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## Abstract

A free lattice fermion field theory in 1+1 dimensions can be interpreted as SOS-type model, whose spins are integer-valued. We point out that the relation between these spins and the fermion field is similar to the abelian bosonization relation between bosons and fermions in the continuum. Though on the lattice the connected  $2n$ -point correlation functions of the integer-valued spins are not zero for any  $n \geq 1$ , the two-point correlation function of these spins is that of free bosons in the infrared. We also conjecture the form of the Wess-Zumino-Witten chiral field operator in a nonabelian lattice fermion model. These constructions are similar in spirit to the “twistable string” idea of Krammer and Nielsen.

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# 1 Introduction

The observation that boson excitations exist in theories of fermions in one space and one time dimension was first made by Tomonaga [1]. Subsequent advances were made for the Luttinger model [2] and the massive Thirring model [3].

The relation of the gradient of a one-component boson field  $\phi$  and the current of a one-flavor dirac fermion field  $\psi$  is

$$: \bar{\psi} \gamma_\mu \psi := \pi^{-\frac{1}{2}} \epsilon_{\mu\nu} \partial^\nu \phi , \quad (1)$$

where  $\mu, \nu = 0$  is the time index and  $\mu, \nu = 1$  is the space index. One can show that the local commutation relations of the operators on each side of equation (1) are the same. This relation is not easily generalized to the lattice where the algebra of the local fermion currents does not close. In fact, one does not expect a precise lattice analogue of (1). The best one can hope for is that a “good” set of mutually commuting operators can be defined in a free lattice fermion theory. Such a set of operators would be good in the sense that their correlation functions become those of free boson fields in the infrared. In other words, the boson correlation functions should flow along renormalization-group trajectories to the correlation functions of the gaussian model.

Integrating equation (1) for  $\mu = 0$  gives

$$\phi(x) =: \sqrt{\pi} \int_0^x dz [\psi^\dagger(z) \psi(z)] : , \quad (2)$$

where one boundary is at  $x = 0$ . The constant of integration is fixed to zero by the requirement that the vacuum expectation value of  $\phi$  is zero. We will show that there is a natural lattice analogue of (2).

A free lattice fermion field theory in 1+1 dimensions can be interpreted as SOS-type model, whose spins are integer-valued. We point out that the relation between these spins and the fermion field is similar to the bosonization relation between bosons and fermions in the continuum. While the connected  $2n$ -point correlation functions of the integer-valued spins are not zero for any  $n \geq 1$ , the two-point correlation function of these spins is that of free bosons in the infrared. This supports the arguments of den Nijs that in the massless phase, SOS-type models are described in the infrared by the gaussian model [4].

We begin by considering the antiferromagnetic XX chain [5], with hamiltonian

$$H = \frac{1}{2} \sum_{n=1}^{N-1} \sum_{\pm} \sigma_n^\pm \sigma_{n+1}^\mp , \quad (3)$$

where the spin operators at each lattice site,  $n$ , are pauli matrices  $\sigma_n^x$ ,  $\sigma_n^y$  and  $\sigma_n^z$ , with  $2\sigma_n^\pm = \sigma_n^x \pm i\sigma_n^y$ .

The XX chain can be viewed as a lattice model of free relativistic fermions. This can be seen by making a Jordan-Wigner transformation [6] to Fermion fields

$$\psi_l = \begin{pmatrix} \alpha_l \\ \beta_l \end{pmatrix} , \quad \psi_l^\dagger = \begin{pmatrix} \alpha_l^\dagger \\ \beta_l^\dagger \end{pmatrix} , \quad (4)$$

where

$$\begin{aligned}\alpha_l^\dagger &= \sigma_{2l+1}^+ \prod_{m=1}^{2l} (-i\sigma_m^z), \quad \alpha_l = \sigma_{2l+1}^- \prod_{m=1}^{2l} (i\sigma_m^z), \\ \beta_l^\dagger &= \sigma_{2l}^+ \prod_{m=1}^{2l-1} (-i\sigma_m^z), \quad \beta_l = \sigma_{2l}^- \prod_{m=1}^{2l-1} (i\sigma_m^z),\end{aligned}\tag{5}$$

which satisfy the local anticommutation relations

$$[\alpha_l^\dagger, \alpha_{l'}]_+ = \delta_{ll'}, \quad [\beta_l^\dagger, \beta_{l'}]_+ = \delta_{ll'},\tag{6}$$

with all other anticommutators equal to zero. It is easy to transform (3) into a dirac hamiltonian with these operators. A naive lattice operator resembling (2) is

$$\phi_n = \sqrt{\pi} \sum_{l=0}^n : \psi_l^\dagger \psi_l := \sqrt{\pi} \sum_{l=0}^n \left( \frac{1}{2} \sigma_{2l}^z + \frac{1}{2} \sigma_{2l+1}^z \right).\tag{7}$$

It is well known that the vacuum expectation value of  $\sigma_n^z$  vanishes [5]. We will show that the natural choice of a lattice bosonic field operator is similar, though not identical to (7). Both operators become (2) in the continuum limit.

It is also true that a spin configuration can be viewed as a “height profile” through the integer-eigenvalued operator

$$h_n = \sum_{l=1}^n \sigma_l^z.\tag{8}$$

This operator strongly resembles  $\phi_l$  defined in (7) (except for a factor of  $\frac{\sqrt{\pi}}{2}$ ). The eigenvalues  $\lambda_n$  of the  $h_n$  are restricted by  $\lambda_{n+1} - \lambda_n = 0, \pm 1$ ,  $\lambda_0 = 0$ . The Hamiltonian (1) is equivalent to a restricted SOS-model hamiltonian

$$H = \frac{1}{2} \sum_{n=1}^{N-1} \sum_{\pm} h_n^\pm h_{n+1}^\mp,\tag{9}$$

where

$$[h_n^\pm, h_m] = \mp h_n^\pm \delta_{nm}, \quad [h_n^+, h_m^-] = 0.\tag{10}$$

In the ground state of the XX chain, the magnetization is zero. Therefore it is clear that the expectation value of  $h_n$  is zero. One can imagine “coarse graining” the spin configurations of the SOS model by some sort of renormalization procedure. Since the model is known to be gapless, it is reasonable to conjecture that in the infrared the height profile becomes a continuous function, described by a free massless boson field theory of a field  $\phi$  which vanishes on the boundaries. One piece of supporting evidence for this assertion is that the central charge of the conformally invariant critical theory can be found by calculating the casimir energy (the connection between the two can be found in [7]). The casimir energy turns out to be identical to that for free bosons with dirichlet boundary conditions.

From the above discussion, it seems that the lattice boson field should be

$$\phi_n = \frac{\sqrt{\pi}}{2} h_n = \frac{\sqrt{\pi}}{2} \sum_{l=1}^n \sigma_l^z.\tag{11}$$

In this letter we will show that this is indeed the case.

By a different jordan-wigner transformation [5]

$$c_n^\dagger = \sigma_n^+ \prod_{m=1}^{n-1} \sigma_m^z, \quad c_n = \sigma_n^- \prod_{m=1}^{n-1} \sigma_m^z, \quad (12)$$

which implies the local anticommutation relations

$$[c_n^\dagger, c_m]_+ = \delta_{nm}, \quad [c_n, c_m]_+ = [c_n^\dagger, c_m^\dagger]_+ = 0, \quad (13)$$

the antiferromagnetic H=hamiltonian becomes a hopping hamiltonian of free fermions

$$H = \sum_k \cos k \, c^\dagger(k) c(k), \quad (14)$$

where  $k = \frac{\pi}{N+1}m$ ,  $m = 1, \dots, N$  and

$$c_n = \left(\frac{2}{N}\right)^{\frac{1}{2}} \sum_k \sin nk \, c(k), \quad c_n^\dagger = \left(\frac{2}{N}\right)^{\frac{1}{2}} \sum_k \sin nk \, c^\dagger(k). \quad (15)$$

In this formulation, the two-point correlation function of  $\phi_n$  in the heisenberg representation is

$$f(n, t; n', t') = \langle 0 | \mathcal{T} \{ \phi_n^H(t) \phi_{n'}^H(t') \} | 0 \rangle = \pi \langle 0 | \mathcal{T} \{ h_n^H(t) h_{n'}^H(t') \} | 0 \rangle, \quad (16)$$

where for any operator  $A$ ,  $A^H(t) \equiv e^{-iHt} A e^{iHt}$ . In terms of the operators (14) this two-point function is

$$f(n, t; n', t') = \frac{\pi}{4} \langle 0 | \mathcal{T} \left\{ \sum_{l=1}^n [2c_l^{\dagger H}(t) c_l^H(t) - 1] \sum_{l'=1}^{n'} [2c_{l'}^{\dagger H}(t') c_{l'}^H(t') - 1] \right\} | 0 \rangle. \quad (17)$$

We will calculate  $f(n, t; n', t')$  for  $t = t'$  exactly and show that for  $N \gg |n - n'| \gg 1$  it has the form of the two-point function of a free massless bosonic field theory, in which the boson field vanishes at the boundaries. For  $t \neq t'$ , the lorentz transformation properties of the low-lying states should imply that the two-point function has the standard form in this case as well. We will give some further arguments supporting this assertion at the end of this letter.

We feel it is important to point out that while (16) is the correlation function of a bosonic theory which becomes free in the infrared, *the theory is not free on the lattice*. Indeed, the higher-point lattice green's functions  $\langle 0 | \mathcal{T} \{ \phi_{n_1}^H(t) \dots \phi_{n_{2M}}^H(t') \} | 0 \rangle$ ,  $M > 1$ , are not zero. It seems clear that the interaction in the bosonic theory is irrelevant and that all these higher-point functions vanish in the infrared. However, we have not yet systematically studied this issue.

The expression (17) can be reduced to

$$f(n, t; n', t') = -\frac{4\pi}{N^2} \sum_{l=1}^n \sum_{l'=1}^{n'} \sum_{k, k'} G(k, t' - t) G(k', t - t') \sin kl \sin kl' \sin k'l \sin k'l', \quad (18)$$

where the minus sign results from fermi statistics and the standard fermion propagator,  $G(k, t - t')$ , is given by

$$\begin{aligned} G(k, t - t') &= \langle 0 | \mathcal{T} \{ c^H(k, t) c^\dagger(k', t') \} | 0 \rangle \\ &= [\theta(t - t') \langle 0 | c(k) c^\dagger(k) | 0 \rangle - \theta(t' - t) \langle 0 | c^\dagger(k) c(k) | 0 \rangle] e^{-i(t-t') \cos k}. \end{aligned} \quad (19)$$

Note that  $\langle 0 | c^\dagger(k) c(k) | 0 \rangle = \theta(-\cos k)$  and  $\langle 0 | c(k) c^\dagger(k) | 0 \rangle = \theta(\cos k)$  (this follows from the fact that the fermi surface is at  $\cos k = 0$ ).

Using

$$\sum_{l=1}^n \sin kl \sin k'l \sum_{l'=1}^{n'} \sin kl' \sin k'l' = \frac{1}{8} \sum_{s=-n}^n \sum_{s'=-n'}^{n'} e^{ik(s+s')} [\cos k'(s+s') - \cos k'(s-s')] . \quad (20)$$

Neglecting a contribution of  $O(1/N^2)$  from the fermi surface gives

$$\begin{aligned} f(n, t; n', t') &= -\frac{\pi}{2N^2} \sum_{s=-n}^n \sum_{s'=-n'}^{n'} \left[ \theta(t' - t) \sum_{m=1}^{\frac{N}{2}} - \theta(t - t') \sum_{m=\frac{N}{2}+1}^N \right] \\ &\times \left[ \theta(t - t') \sum_{m'=1}^{\frac{N}{2}} - \theta(t' - t) \sum_{m'=\frac{N}{2}+1}^N \right] e^{i \frac{\pi}{N+1} (s+s')m} \\ &\times \left[ \cos \frac{\pi}{N+1} (s+s')m - \cos \frac{\pi}{N+1} (s-s')m' \right] \\ &\times \exp[i(t-t')(\cos \frac{\pi m}{N+1} - \cos \frac{\pi m'}{N+1})] . \end{aligned} \quad (21)$$

At equal times,  $t = t'$ , this expression simplifies further, so that we may carry out the summation over  $m$  and  $m'$ . Replacing  $N+1$  by  $N$  introduces errors of order  $1/N$  and will not affect (21) in the thermodynamic limit. The result is

$$\begin{aligned} f(n, n') &\equiv f(n, t; n', t) = \frac{i\pi}{2N^2} \sum_{s=-n}^n \sum_{s'=-n'}^{n'} \frac{\sin^2 \frac{\pi}{4} (s+s') e^{i \frac{\pi}{2} (s+s')}}{\sin \frac{\pi}{2N} (s+s')} \\ &\times \left[ \frac{\sin^2 \frac{\pi}{4} (s+s') \sin \frac{\pi}{2} (s+s')}{\sin \frac{\pi}{2N} (s+s')} - \frac{\sin^2 \frac{\pi}{4} (s-s') \sin \frac{\pi}{2} (s-s')}{\sin \frac{\pi}{2N} (s-s')} \right] . \end{aligned} \quad (22)$$

The non-vanishing terms in the sum on the right-hand-side of (22) are those for which  $s+s'$  is odd. For large  $N$  we may replace this sum by an integral, dividing by a factor of two (because half of the terms in the sum vanish). Defining  $x = \frac{\pi s}{2N}$  and  $x' = \frac{\pi s'}{2N}$ , the correlation function is

$$f(n, n') = \frac{1}{8\pi} \int_{-\frac{\pi n}{2N}}^{\frac{\pi n}{2N}} dx \int_{-\frac{\pi n'}{2N}}^{\frac{\pi n'}{2N}} dx' \left[ \frac{1}{\sin^2(x+x')} - \frac{1}{\sin(x+x') \sin(x-x')} \right] . \quad (23)$$

Evaluation of the integral yields

$$f(n, n') = \frac{1}{2\pi} \left[ \ln \left| \frac{\sin \frac{\pi}{2N} (n+n')}{\sin \frac{\pi}{2N} (n-n')} \right| - Li_2 \left( \frac{\tan \frac{\pi n}{2N}}{\tan \frac{\pi n'}{2N}} \right) + Li_2 \left( -\frac{\tan \frac{\pi n}{2N}}{\tan \frac{\pi n'}{2N}} \right) \right] , \quad (24)$$

where the polylogarithm function  $Li_2(z)$  is given by  $Li_2(z) = \sum_{r=1}^{\infty} \frac{z^r}{r^2}$  [8].

It is simple to take the continuum limit of (24). If we set  $r = na$  and  $r' = n'a$ , where  $a$  is the lattice spacing and take  $N \rightarrow \infty$ , with volume  $L = Na$  fixed, the two-point function of bosonic fields is

$$f(r, r') = \frac{1}{2\pi} \left[ \ln \left| \frac{\sin \frac{\pi}{2L}(r + r')}{\sin \frac{\pi}{2L}(r - r')} \right| - Li_2\left(\frac{\tan \frac{\pi r}{2L}}{\tan \frac{\pi r'}{2L}}\right) + Li_2\left(-\frac{\tan \frac{\pi r}{2L}}{\tan \frac{\pi r'}{2L}}\right) \right], \quad (25)$$

which vanishes for  $r = 0$  or  $r' = L$ .

For  $L \gg |r - r'|$ , the arguments of the polylogarithm functions tend to plus one and minus one. Therefore the last two terms merely give a constant contribution:

$$Li_2\left(\frac{\tan \frac{\pi r}{2L}}{\tan \frac{\pi r'}{2L}}\right) + Li_2\left(-\frac{\tan \frac{\pi r}{2L}}{\tan \frac{\pi r'}{2L}}\right) \longrightarrow \sum_{r=1}^{\infty} \frac{1 - (-1)^r}{r^2} = \frac{\pi^2}{4}, \quad (26)$$

so

$$f(r, r') \approx -\frac{1}{2\pi} \ln \left| \sin \frac{\pi}{2L}(r - r') \right| + \frac{1}{2\pi} \ln \left| \sin \frac{\pi}{2L}(r + r') \right| - \frac{\pi}{8}, \quad (27)$$

which is the massless scalar propagator for dirichlet boundary conditions with an expected non-universal constant added. Such constants always appear with lattice regularizations (see for example Spitzer [9]).

We have only calculated the equal-time two-point function of  $\phi_n$ . There is a simple argument (which carries more weight than that based on lorentz properties of the spectrum) that the time-ordered correlation function of two field operators at the same lattice site but at different times should also be that of free bosons in the infrared. The ground state of the XX chain coincides with that of the six-vertex model with parameter  $\Delta$  equal to zero [10]. The six-vertex model has an SOS interpretation similar to that of the spin chain [4]. Since the boltzmann factor of this model has  $90^\circ$  rotation invariance, its correlation functions should have this invariance also. In the continuum limit, the six-vertex model coincides with the XX chain, so the two-point correlation functions of the latter will also be  $90^\circ$  rotation invariant. We suspect that the two-point function can be calculated in the case of arbitrary space and time separation.

We have attempted to make a similar construction on the lattice for nonabelian bosonization [11]. In particular, we have found a candidate for the chiral boson field operator of the Wess-Zumino-Witten model, for global symmetry group  $spin(4) \otimes U(1)$ . We have not evaluated any of the correlation functions of this field as yet.

If  $\psi^A$ ,  $A = 1, \dots, 4$  is a  $1 + 1$ -dimensional dirac field, the Wess-Zumino-Witten field,  $g \in spin(4) \otimes U(1)$ , is given by

$$\bar{\psi}^A \gamma_\mu \psi^B = i \frac{1}{2\pi} \epsilon_{\mu\nu} [g \partial^\nu g^{-1}]^{AB}, \quad (28)$$

Integrating equation (28) for  $\mu = 0$  gives the equation analogous to (2)

$$g(x) = \mathcal{P} \exp 2\pi i \int_0^x dz \psi^\dagger(z) \psi(z), \quad (29)$$

where  $\mathcal{P}$  denotes path ordering, one boundary is again at  $x = 0$  and the bilinear  $\psi^\dagger(z) \psi(z)$  is understood as a  $4 \times 4$  matrix.

There is a lattice spin chain which is equivalent to free fermions with the symmetry group  $spin(4) \otimes U(1)$  [12], [13]. Its hamiltonian is

$$H = \frac{1}{4} \sum_{n=1}^{N-1} \{ [1 + (-1)^n] \gamma_n^A \gamma_{n+1}^A + [1 - (-1)^n] \rho_n^A \rho_{n+1}^A \} , \quad (30)$$

where the operators at each site  $n$  are defined by

$$[\gamma_n^A, \gamma_n^B]_+ = 2\delta^{AB} , \quad \gamma_n^5 = \gamma_n^1 \gamma_n^2 \gamma_n^3 \gamma_n^4 , \quad \rho_n^A = -i \gamma_n^5 \gamma_n^A , \quad \sigma_n^{AB} = \frac{-i}{4} [\gamma_n^A, \gamma_n^B] , \quad (31)$$

which constitute a basis for the lie algebra of  $SU(4)$ . The operators  $\gamma_n^A$  should not be confused with the matrices  $\gamma_\mu$ . The 6 generators of  $spin(4)$  are  $\sum_n \sigma_n^{AB}$  while the generator of  $U(1)$  is  $\sum_n \gamma_n^5$ .

A jordan-wigner transformation [12], [13] analogous to (4), (5) is

$$\psi_l^A = \begin{pmatrix} \alpha_l^A \\ \beta_l^A \end{pmatrix} , \quad (32)$$

where

$$\alpha_l^A = \frac{1}{\sqrt{2}} \gamma_{2l+1}^A \prod_{m=1}^{2l} \gamma_m^5 , \quad \beta_l^{A\dagger} = \frac{1}{\sqrt{2}} \rho_{2l}^A \prod_{m=1}^{2l-1} \gamma_m^5 , \quad (33)$$

which satisfy the local anticommutation relations

$$[\alpha_l^A, \alpha_{l'}^B]_+ = \delta^{AB} \delta_{ll'} , \quad [\beta_l^A, \beta_{l'}^B]_+ = \delta^{AB} \delta_{ll'} , \quad (34)$$

with all other anticommutators equal to zero. This transformation converts (30) into a free massless hamiltonian of a majorana-dirac field with 4 real components.

The lattice operator-valued matrix analogous to the continous operator-valued matrix  $\psi_A^\dagger(z) \psi_B(z)$  is

$$\Omega_n^{AB} = \gamma_n^A \gamma_n^B = \rho_n^A \rho_n^B = \delta^{AB} + 2i \sigma_n^{AB} . \quad (35)$$

On the basis of (29) we conjecture that the correlation functions of

$$g_n = \prod_{m=1}^n \exp 2\pi i \Omega_m , \quad (36)$$

are those of the Wess-Zumino-Witten model in the infrared.

Some time ago Krammer and Nielsen proposed the idea that a “twistable string” [14], with only bosonic degrees of freedom on the world-sheet was equivalent to a Neveu-Schwartz-Ramond string. These authors were motivated by the construction in reference [12] and argued (using methods quite different from ours) that one could describe bosons as well as fermions using  $(1+1)$ -dimensional spin densities.

In summary, we have found the operator in a  $1+1$ -dimensional theory of free relativistic fermions whose two-point function is that of free bosons in the infrared. This operator is proportional to the integer-valued spin in an SOS formulation of the fermionic hamiltonian. Finally we have conjectured a similar bosonization relation for a nonabelian theory of relativistic fermions.

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